MA6452-PROBABILITY AND QUEUEING THEORY

UNIT I: RANDOM VARIABLES
PART A

1. Mathematical or Apriori Definition Of Probability.

Let S be the sample space and A be an event associated with a random experiment. Let n(s) and n(A) be the number of elements of 'S' and 'A'. Then the probability of an event 'A' occurring, denoted by P(A), is defined by P(A) = n(A) /n(S) = No. of favorable cases / Total No. of Possible cases.

2. State axioms of probability

If A be any event associated with a random experiment whose sample space is S, then probability of the event A, P(A) satisfies the following axioms (i) $0 \le P(A) \le 1$ (ii) P (S) =1 (iii) If $A_1, A_2, \dots A_n$ are mutually exclusive events then $P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots P(A_n)$

3. From a bag containing 3 red and 2 black balls, two balls are drawn at random. Find the probability that they are of the same color.

P (same colour) = $\frac{3C_2 + 2C_2}{5C_2} = \frac{3+1}{\frac{5*4}{1*2}} = \frac{4}{10} = \frac{2}{5}$

4. When A and B are 2 mutually exclusive events such that P (A) = $\frac{1}{2}$ and P (B) = $\frac{1}{3}$ find $P(A \cup B)$ and $P(A \cap B)$.

When A and B are mutually exclusive $P(A \cup B)$ = P (A) + P (B) = $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

$P(A \cap B) = 0$

5. If P (A) = 0.29 and P (B) = 0.43 find $P(A \cap \overline{B})$, if A and B are mutually exclusive

If A and B are mutually exclusive $P(A \cap B) = 0$,

$P(A \cap \overline{B})$ = P (A) - $P(A \cap B)$ = P (A) – 0 = 0.29

6. When A and B are mutually exclusive events, are the values P (A) = 0.6 and $P(A \cap \overline{B})$ = 0.5 consistent? Why?

If A and B are mutually exclusive $P(A \cap B) = 0$,

$P(A \cap \overline{B})$ = P (A) - $P(A \cap B)$ = P (A) – 0 = P (A)

But here $P(A \cap \overline{B}) \neq$ P (A), so given values are inconsistent.

7. If P (A) = $\frac{3}{4}$ P(B) = $\frac{5}{8}$, prove that $P(A \cap B) \ge \frac{3}{8}$

$P(A \cup B) \le 1 \Rightarrow P(A) + P(B) - P(A \cap B) \le 1$

$$\frac{3}{4} + \frac{5}{8} - 1 \le P(A \cap B) \Rightarrow P(A \cap B) \ge \frac{3}{8}$$

8. If P (A) = 0.4 P (B) = 0.7 and $P(A \cap B)$ = 0.3 find $P(\overline{A} \cap \overline{B})$

$P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B})$ = 1 - $P(A \cup B)$ = 1 – [0.4 + 0.7 -0.3] = 0.2

9. If A and B are mutually exclusive events prove that P (B/A) = 0

If A and B are mutually exclusive, $P(A \cap B)$ = 0

P (B/ A) = $\frac{P(A \cap B)}{P(B)}$ = $\frac{0}{P(B)}$ = 0

10. If P (A) = 1/3 P (B) = 3/4 and $P(A \cup B)$ = 11/12, find P (A/B) and P (B/A)

$P(A \cup B)$ = P (A) + P (B) – P (A ∩ B)

P (A ∩ B) = 1/3 +3/4 -11/12 = 2/12 = 1/6

P (A /B) = $\frac{P(A \cap B)}{P(A)}$ = $\frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{6} * \frac{3}{1} = \frac{1}{2}$ P (B/ A) = $\frac{P(A \cap B)}{P(B)}$ = $\frac{\frac{1}{6}}{\frac{3}{4}} = \frac{1}{6} * \frac{4}{3} = \frac{2}{9}$

11. If P (A) = 0.65 and P (B) = 0.4 and P (A ∩ B) = 0.24, can A and B be independent

If A and B are independent then P (A ∩ B) = P (A). P (B).

Here P (A). P (B) = 0.65 * 0.4 = 0.140 ≠ 0.24.So A and B are not independent.

12. If P (A) = 0.5 and P (B) = 0.3 and P (A ∩ B) = 0.15, find $P(A/\overline{B})$

$P(A/\overline{B})$ = $\frac{P(A \cap \overline{B})}{P(\overline{B})} = \frac{P(A) - P(A \cap B)}{1 - P(B)}$ = $\frac{0.5 - 0.15}{1 - 0.3} = \frac{0.35}{0.7} = 0.5$

13. If A and B are independent then prove that $P(A \cup B)$ = 1 - $P(\overline{A}) * P(\overline{B})$

$P(A \cup B)$ = 1 - $P(\overline{A \cup B})$ = 1 - $P(\overline{A} \cap \overline{B})$ = 1 - $P(\overline{A}) * P(\overline{B})$

14. Conditional probability:

The conditional probability of an event B, assuming that event A has happened is denoted by P (B/A) defined as P(B/A) = n(A∩B) / n(A) = P(A∩B) / P(A), Provided P(A) ≠ 0

15. Multiplication theorem of probability:

For any two events A and B P (A∩B) = P (A). P (B/A); Provided P (A) ≠ 0

P (A∩B) = P (B). P (A/B): Provided P (B) ≠ 0

For two independent events P (A∩B) = P (A). P (B)

16. A and B are two events with P(A) = 3/8 P(B) = ½ and P(A∩B) = ¼ . Find P (A^C ∩B^C).

P (A^C ∩B^C) = P ($(A \cup B)^C$)

$= 1 - P(A \cup B)$
$= 1 - [P(A) + P(B) - P(A \cap B)]$
$= 1 - [3/8 + ½ - ¼]$
$= 3/8$

17. A and B are events such that $P(A \cup B) = ¾$ $P(A \cap B) = ¼$ $P(A^C) = 2/3$. Find $P(A^C/B)$

$P(A^C/B) = P(A^C \cap B)/P(B)$

$P(A) = 1 - P(A^C) = 1 - 2/3 = 1/3$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$3/4 = 1/3 + P(B) - ¼$
$P(B) = 2/3$

$P(A^C \cap B) = P(B) - P(A \cap B)$
$= 2/3 - ¼ = 5/12$

$P(A^C/B) = 5/8$

18. If you twice flip a coin what is the probability that getting at least one head

S = {HH, HT, TH, TT} Let E be the event of getting at least one head. E = {HH, HT, TH}
P (E) = n (E) /n(S) = ¾

19. If the probability that a communication system has high selectivity is 0.54 and the probability that it will have high fidelity is 0.81 and the probability that it will have both is 0.18. Find the probability that a system with high fidelity will also have high selectivity?

Let A be the event of having a system with high selectivity and B be the event with high fidelity. Given P (A) = 0.54 P(B) = 0.81 $P(A \cap B)$ = 0.18
P(a system with high fidelity will also have high selectivity) = $P(A/B) = P(A \cap B) / P(A) = 0.18 / 0.81 = 0.22$

20. The probability that a new airport will get an award for its design is 0.16. The probability that it will get an award for the efficient use of materials is 0.24 and the probability that it will get both awards is 0.11. What is the probability that it will get only one of two awards?

Let D be the event of getting award for its design and E be the event of getting award for its efficient. Given P (D) = 0.16 P (E) = 0.24 $P(D \cap E)$ = 0.11

P (Getting only one of the two awards) = $P(D^C \cap E) + P(D \cap E^C)$
$= P(E) - P(D \cap E) + P(D) - P(D \cap E)$
$= 0.24 - 0.11 + 0.16 - 0.11 = 0.18$

21. There are 3 unbiased coins and 1 biased coin with head on both sides. A coin is chosen at random and tossed 4 times. If head occurs all the 4 times what is the probability that the biased coin has been chosen?

Let A_1 be the event of choosing a coin which is biased and A_2 be the event of choosing a unbiased coin. $P(A_1) = 1/4$ $P(A_2) = 3/4$. Let B be the event of getting 4 heads when the randomly chosen coin 4 times. $P(B/A_1) = 1$ $P(B/A_2) = 4C_4 (1/2)^4 (1/2)^0 = 1/16$

$P(B) = P(A_1) P(B/A_1) + P(A_2) P(B/A_2) = 1/4 . 1 + 3/4 . 1/16 = 0.3125$

22. Given the probability density function of a continuous random variable X as follows f(x) = 6x (1-x) 0<x<1 . Find cumulative density function.

$$\text{CDF } F(x) = \int_0^x f(x)\,dx = \int_0^x 6x(1-x)\,dx = 6(x^2/2 - x^3/3)\Big|_0^x = 3x^2 - 2x^3, \quad 0 < x < 1$$

23. A continuous random variable X can assume any value between x = 2 and x = 5 has a density function given by f(x) = k(x + 1). Find P(X < 4)

$$\int_2^5 f(x)\,dx = 1 \Rightarrow \int_2^5 k(x+1)\,dx = k(x^2/2 + x)\Big|_2^5 \Rightarrow k(27/2) = 1 \Rightarrow k = 2/27.$$

$$P(X < 4) = \int_2^4 f(x)\,dx = \int_2^4 k(x+1)dx = 2/27\int_2^4 (x+1)\,dx = 2/27\,(x^2/2 + x)\Big|_2^4 = 16/27$$

24. Four persons are chosen at random from a group containing 3 men, 2 women and 4 children. Show that the chance that exactly two of them will be children is 10/21

Let A be the event of selecting 2 children 1 woman and 1 man.

$n(A) = 4C_2 . 2C_1 3C_1$

Let B be the event of selecting 2 children 2 men

$n(B) = 4C_2 . 3C_2$

Let C be the event of selecting 2 children 2 women

$n(C) = 4C_2 . 2C_2$

P(Exactly 2 children) = $P(A \cup B \cup C) = P(A) + P(B) + P(C) = 10/21$

25.If moment generating function $M_X(t) = 1/3\, e^t + 4/15\, e^{3t} + 2/15\, e^{4t} + 4/15\, e^{5t}$. Find the probability mass function of X.

X	1	2	3	4	5
P(X = x)	1/3	0	4/15	2/15	4/15

26. Suppose $M_X(t) = (0.4e^t + 0.6)^8$. Find the MGF of Y = 3X + 2

$M_Y(t) = e^{2t} M_X(3t) = e^{2t} (0.4e^{3t} + 0.6)^8$

27. Define MGF. Why it is called so?

MFG is defined as $M_X(t) = E(e^{tx})$, which generates the moments of the random variable X, so it is called as Moment Generating function.

28. For a binomial distribution mean is 6 and standard deviation is √2. Find the first two terms of the distribution.

Soln:

For a binomial distribution,

Mean= np = 6. S.D. = √npq

npq=2

6 X q =2, q= 1/3. P=

1-q = 1- 1/3 = 2/3.

n = 9.

$P(X=x) = nC_x\ p^x\ q^{n-x}$, x = 0, 1, 2, ...,n

$P(X=0) = 9C_0\ (2/3)^0\ (1/3)^9 = 0.0005$

$P(X=1) = 9C1\ (2/3)\ (1/3)^8 = 0.0009$.

29. Comment of the following.

" The mean of a binomial distribution is 3 and variance is 4" For a binomial distribution, variance.

Soln:

np = 3, npq = 4, (3) q=
4, q = (4/3). It is not
possible.
∴ The given statement is wrong.

30. Find p for a binomial variate, if n=6 and 9 P(X- 4) = P(X =2)

Soln:

$P(X=x) = 6C_x\ p^x\ q^{6-x}$, 9P(X = 4) = P(X = 2),

$9 \times 6C_4\ p^4\ q^2 = 6C_2\ p^2\ q^4$

$9\ p^2 = q^2$,

$9\ p^2\ (1-p)^2$, since q= 1- p.

$9p^2 = 1 + p^2 - 2p$

$8p^2 + 2p - 1 = 0$,

[p+ (1/2)] [p – (1/4)] =0, P =
¼.

31. State the conditions under which the poisson distribution is a limiting case of the Binomial Distribution.

Soln:

(i) n, the number of trials infinitely large, $n \to \infty$ (ii) p, the prob. Of success infinitely small, $p \to 0$ (iii) $np = \lambda$, a constant.

32. In a book of 520 pages, 390 typo- graphical errors occur. Assuming poisson law for the number of errors per page, find the prob. That a random sample of 5 pages wills certain no error.

Soln:

$\lambda = 390 / 320 = 0.75$,

$P(X= x) = e^{-\lambda} \lambda^{x} / x!$, x= 0, 1, 2, ...

$= e^{-0.75} (0.75)^{x} / x!$

Required prob. $= (P[X = 0])^{2}$,

$= (e^{-0.75})^{2}$,

$= e^{-3.75}$

33. If X is the number of occurrences of the Poisson variate with mean λ, show that P[X ≥ n] – P[X ≥ n+1] = P[X – n].

Soln:

$P[X \geq n] - P[X \geq n+1] = 1 - P[X < n] - (1 - P[X < n+1])$

$= - P[X < n] + P[X < n+1]$

$$= \sum_{x=0}^{n} \{ e^{-\lambda} \lambda^{x} / x! \} - \sum_{x=0}^{n-1} \{ e^{-\lambda} \lambda^{x} / x! \}$$

$= e^{-\lambda} \lambda^{n} / n! = P[X=n]$.

34. If X is poisson variate such that P[X=2] = (2/3) P[X = 1] evaluate

P[X=3].

$e^{-\lambda} \lambda^{2} / 2! = (2/3) \{e^{-\lambda} \lambda / 1! \}$

$\lambda = 4 / 3$.

$P[X=3] = e^{-(4/3)} (4/3)^{3} / 3!$

35. If for a poisson variate X, $E(X^{2}) = 6$ what is E(X).

$E(X^{2}) = \lambda^{2} + \lambda$, since poisson distribution $\mu_2 = \lambda^{2} + \lambda$

Given $E(X^{2}) = 6$,

$\lambda^2 + \lambda = 6$

$\lambda^2 + \lambda - 6 = 0$

$\lambda = 2, -3$

But $\lambda > 0$, $\lambda = 2$.

Therefore E(X) = 2.

36. If X is a poisson variate with mean λ, Show that $E(X^2) = \lambda E(X+1)$.

W. K. T. For a poisson variate

$E(X^2) = \lambda^2 + \lambda$

Now $E(X+1) = E(X) + E(1) = E(X) + 1$

$= \lambda + 1$

Now $E(X^2) = \lambda^2 + \lambda$

$= \lambda(\lambda + 1)\ E(X^2)$

$= \lambda E(X+1)$

37. Determine the distribution whose M.G.F is $M_X(t) = e^{3(e\,t - 1)}$. Also

Find P(X=1).

W.K.T. M.G.F of a poisson distribution

$M_X(t) = e^{\lambda(e\,t - 1)}$

Given $M_X(t) = e^{3(e\,t - 1)}$

$\lambda = 3$

$P(X = x) = e^{-\lambda} \lambda^x / x!$, x =0, 1, 2, ...

$P(X = x) = e^{-3}\, 3^x / x!$, x =0, 1, 2, ...

$P(X = 1) = e^{-3}\, 3 / 1!$, x =0, 1, 2, ...

$= 3e^{-3}$

38. Find the M.G.F of geometric distribution.

$M_X(t) = E(e^{tx})$

$$= \sum_{x=0}^{\infty} e^{tx}.\ P(X=x)$$

$$= \sum_{x=0}^{\infty} e^{tx}.\ p\, q^x$$

∞

$= p \sum_{x=0} (e^t q)^x.$

$= p [1 - q e^t]^{-1}$

$= p / (1 - q e^t)$

39. Identify the distribution with MGF $M_X(t) = (5 - 4e^t)^{-1}$

$M_X(t) = \{ (1/5) / [1 - (4/5)e^t] \}$

WKT, MGF of Geometric distribution, $M_X(t) = p / (1 - q e^t)$

∴ The Given MGF is the MGF of geometric distribution with parameter.

$P = 1/5$, $q = 4/5$.

$P[X = x] = (1/5)(4/5)^{x-1}$, $x = 1, 2, \ldots\ldots$

40. Determine the distribution whose MGF is $M_X(t) = (1/3) e^t [e^{-t} - (2/3)]^{-1}$

Soln:

The $M_X(t)$ of negative binomial distribution with parameters p and r is

$M_X(t) = p^r [1 - q e^t]^{-r}$,

Given $M_X(t) = (1/3) [e^t e^{-t} - (2/3) e^t]^{-1}$,

$= (1/3) [1 - (2/3) e^t]^{-1}$,

This is the M.G.F of Negative Binomial Distribution with parameters r = 1 and p = 1/3.

41. Find the MGF of a uniform distribution in (a, b)

$M_X(t) = [1 / (b-a)] = \int^{b} e^{tx}\, x.\, b$

$= [1 / (b-a)] \{e^{tx} / t\} \quad = [e^{bt} - e^{at}] / (b-a)t$

42. Find the MGF of a R.V. which is uniformly distributed over (-1, 2)

Soln:

2

$$M_X(t) = [1/3] = \int_{-1} e^{tx}\, dx. = [(e^{2t} - e^{-t}) / 3t], \; t \neq 0$$

$$M_X(t) = [1/3] = \int_{-1}^{2} dx. = 1, \text{ for } t \neq 0$$

43. If X has uniform distribution in (-3, 3) find P[|X - 2| < 2]

P.d.f is $f(x) = 1/6, -3 < X < 3$

$P[|X - 2| < 2] = P[0 < X < 4]$

$$= (1/6)\int_{0}^{3} dx = 1/2.$$

44. State the additive property of Gamma distribution.

If $X_1, X_2, \ldots X_n$ are independent Gamma variates with parameters $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$ respectively, then $X_1 + X_2 + \ldots + X_n$ is also a Gamma variate with parameter $\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n$.

45. Mention any four properties of normal distribution.

(1) The curve is bell shaped

(2) Mean, median, mode coincide

(3) All odd central moments vanish

(4) X axis is an asymptote toe the normal curve

46. If for a normal distribution, the first moment about 10 is 40 and the fourth moment about 50 is 48. Find the mean and S.D. of the distribution.

Mean = $A + \mu_1'$

Mean = 10 + 40 = 50.

μ_4 (about the point X = 50) = 48.

Since mean is 50,

$3\sigma^2 = \mu_4'$

$3\sigma^2 = 48$; $\sigma = 2$

47. X is a normal variate with mean 1 and variance 4, Y is another normal variate independent of X with mean 2 and variance 3, what is the distribution of X+2Y= ?

E[X+2Y] = E[X] + 2E[Y] = 1 + 4 = 5

V[X+2Y] = V[X] + 4V[Y] = 4 + 4(3) = 16

∴ X + 2 Y is a normal distribution with parameter 5, 16.

48. If the continuous R.V X has p.d.f. f(x)= 2(x+1) / 9, -1 < X < 2. Find the p.d.f. of Y = X^2.

$f_Y(y) = 2 / [9\sqrt{y}]$, 0< y < 1.

$f_Y(y) = 1 / a[1 + (1/\sqrt{y})]$, 1< y < 4.

49. If X is uniformly distributed in (-1, 1). Find the p.d.f. of y = sin [πx / 2]

$f_X(x) = 1 / 2$, -1< x < 1. dy

/dx = cos [πx/2] . π/2

dx / dy = 2 / [$\pi\sqrt{1- y^2}$] , -1 ≤ y ≤1 $f_Y(y)$

= 2 / [$\pi\sqrt{1- y^2}$], -1< y < 1.

50. If X has negative binomial distribution with parameters (n,p) Prove that $M_X(t)$ is (Q – P e^t) $^{-n}$

$$M_X(t) = E(e^{tx}) = \sum_0^\infty nC_x Q^{-n}\left(\frac{-Pe^t}{Q}\right) = (Q - Pe^t)^{-n}$$

51. Find an expression for rth moment of weibull distribution.

$$\mu_r' = E\left[X^r\right] = \alpha\beta\int_0^\infty x^{r+\beta-1}e^{-\alpha x^\beta}\,dx$$ Putting y = αx^β

$$\mu_r' = \alpha^{-\frac{r}{\beta}}\int_0^\infty y^{\frac{r}{\beta}}e^{-y}dy = \alpha^{-\frac{r}{\beta}}\,\Gamma\left(\frac{r}{\beta}+1\right)$$

52. Mention any four properties of Normal distribution

(i) The curve is bell shaped. (ii) Mean, median, mode coincide.

(iii) All odd central moments vanish (iv) X axis is an asymptote to the normal curve.

PART B QUESTIONS

1. A random variable X has the following probability distribution

X	0	1	2	3	4	5	6	7
P(X)	0	K	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

Find (i) The value of k, (ii) P[1.5 < X < 4.5 / X >2] and (iii) The smallest value of λ for which P(X ≤ λ) < (1/2).

2. A bag contains 5 balls and its not known how many of them are white. Two balls are drawn at random from the bag and they are noted to be white. What is the chance that the balls in the bag all are white.

3. Let the random variable X have the PDF f(x) = $\frac{1}{2}e^{-\frac{x}{2}}$, x >0 Find the moment generating function, mean and variance.

4. A die is tossed until 6 appear. What is the probability that it must tossed more than 4 times.

5. A man draws 3 balls from an urn containing 5 white and 7 black balls. He gets Rs. 10 for each white ball and Rs 5 for each black ball. Find his expectation.

6. In a certain binary communication channel, owing to noise, the probability that a transmitted zero is received as zero is 0.95 and the probability that a transmitted one is received as one is 0.9. If the probability that a zero is transmitted is 0.4, find the probability that (i) a one is received (ii) a one was transmitted given that one was received

7. Find the MGF and r^{th} moment for the distribution whose PDF is f(x) = k e^{-x}, x >0. Find also standard deviation.

8. The first bag contains 3 white balls, 2 red balls and 4 black balls. Second bag contains 2 white, 3 red and 5 black balls and third bag contains 3 white, 4 red and 2 black balls. One bag is chosen at random and from it 3 balls are drawn. Out of 3 balls, 2 balls are white and 1 is red. What are the probabilities that they were taken from first bag, second bag and third bag.

9. A random variable X has the PDF f(x) = 2x, 0 < x < 1 find (i) P (X < ½) (ii) P (¼ < X < ½) (iii) P (X > ¾ / X > ½)

10. If the density function of a continuous random variable X is given by

$$f(x) = \begin{cases} ax & 0 \le x \le 1 \\ a & 1 \le x \le 2 \\ 3a - ax & 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(1) Find a (2) Find the cdf of X

11. If the the moments of a random variable X are defined by E (X^r) = 0.6, r = 1,2,.. Show that P (X =0) = 0.4 P (X = 1) = 0.6, P (X ≥ 2) = 0.

12. In a continuous distribution, the probability density is given by f(x) = kx (2 – x) 0 < x < 2. Find k, mean , varilance and the distribution function.

13. The cumulative distribution function of a random variable X is given by

$$\begin{cases} 0, & x < 0 \\ x^2, & 0 \le x \le ½ \end{cases}$$

$$F(x) = 1 - \frac{3}{25}(3-x)^2 \qquad ½ \le x \le 3$$

$$1 \qquad x \ge 3$$

Find the pdf of X and evaluate P (|X| ≤ 1) using both the pdf and cdf

14. Find the moment generating function of the geometric random variable with the pdf $f(x) = p\, q^{x-1}$, x = 1,2,3.. and hence find its mean and variance.

15. A box contains 5 red and 4 white balls. A ball from the box is taken our at random and kept outside. If once again a ball is drawn from the box, what is the probability that the drawn ball is red?
16. A discrete random variable X has moment generating function $M_X(t) = \left(\frac{1}{4}+\frac{3}{4}e^t\right)^5$ Find E(x), Var(X) and P (X=2)
17. The pdf of the samples of the amplitude of speech wave foem is found to decay exponentially at rate α, so the following pdf is proposed f(x) = $Ce^{-\alpha|x|}$, $-\infty < X < \infty$. Find C, E(x)

18. Find the MGF of a binomial distribution and hence find the mean and variance.
19. Find the recurrence relation of central moments for a binomial distribution.

20 The number of monthly breakdowns of a computer is a RV having a poisson distribution with mean equal to 1.8. Find the probability that this computer will function for a month (a) without a breakdown, (b) Wish only one breakdown, (c) Wish at least one break down.
21. Find MGF and hence find mean and variance form of binomial distribution.

22. State and prove additive property of poisson random variable.

23. If X and Y are two independent poisson random variable, then show that probability distribution of X given X+Y follows binomial distribution.
24. Find MGF and hence find mean and variance of a geometric distribution.

25. State and prove memory less property of a Geometric Distribution.

26. Find the mean and variance of a uniform distribution.

27. Find the MGF and hence fine mean and variance of exponential distribution.
28. Find the MGF and hence fine mean and variance of normal distribution.

29. Define exponential distribution and prove the memory less property.

30. In a component manufacturing industry, there is a small probability of

1/500, for any component to be defective. The components are supplied in packets of 10. Use poisson distribution of Calculate the approximate number of packets containing (A) no defective, (B) Two defective components in a consignment of 10,000 packets.

31. The time(in hours required to repair machine is exponentially distributed with parameter $\lambda = ½$. What is the probability that a repair time exceeds 2 hours? What is the conditional probability that a repair time takes at least 10 hours given that its duration exceeds 9 hours.

32. If X is a binomially distributed randm variable with E(X) = 2 and Var. X =

4/3. Find P(X = 5)

33. If X is uniformly distributed in (-2, 2) find (i) P[X < 0], (ii) P [| X - 1| ≥ (1/2)]

34. A random variable X is uniformly distribution over (0, 2π). If Y= cos X, Find

(i) p.d.f. of X (ii) E(X)

35. A pair of dice is thrown 729 times. How many times do you except at least three dice to show 5 or 6.

36. The local authorities in a certain city install 200 electric lamps in a street of a city. If the lamps have an average life of 100 burning hours with a S.D of 2000hours.

(i) What number of lamps might be expected to fail in first 700 hours?

(ii) After what period of burning would be expect that 10 % of the lamps would have failed? Assume that the lives of lamps are normally distribution.

37 If a boy is throwing stones at a target, what is the probability that his 10^{th} throw is his 5^{th} hit, if the probability of hitting the target at any trial is 0.5?

38 Each of the 6 tubes of a radio set has the life length (in years) which may be considered as a random variable that follows a Weibull distribution with parameters $\alpha = 25$ and $\beta = 2$. If these tubes function independently on one another, what is the probability that no tube will have to be replaced during the first 2 months?

39. If the life X (in years) of a certain type of car has weibull distribution with the parameter $\beta = 2$, find the value of the parameter α, given the probability that the life of the car exceeds 5 years is $e^{-0.25}$. For these values of the parameter find the mean and variance.

UNIT- II TWO DIMENSIONAL RANDOM VARIABLES
TWO MARKS

1. **State the basic properties of joint distribution of (X,Y) where x and Y are random variables.**

$F(x,y) = P[X \le x, Y \le y]$. $F(-\infty,\infty) = 1$; $F(-\infty,y)$= Marginal distribution of Y and $F(x,\infty)$ = Marginal distribution of X

2. **The joint p.d.f. f(X,Y) is $f(x,y)= 6e^{-2x-3y}$, $x \ge o$, $y\ge0$, find the conditional density of Y given X.**

<u>Soln.</u>

Marginal of X is $f(x) = \int_0^\infty 6e^{-2x} e^{-3y}\, dy$,

$= 2^{e-2x}$, x>0

Conditiona of Y given x :

$f(y/x) = f(x/y) / f(x)$

$= (6e^{-2x} e^{-3y})/ 2e^{-2x}$

$= 3 e^{-3x}$, $y\ge0$.

3. **State the basic properties of joint distribution of (x,y) when X and Y are random variables.**

<u>Soln.</u>

$F(x,y) = P(X\le x, Y\le y)$

$F(\infty\ \infty) = 1$

$F(\infty, y)$ = Marginal distribution of Y

$F(x, \infty)$ = Marginal distribution of X.

$F(-\infty, y) = 0$, $F(x, -\infty)=0$.

4. **If 2 random variables have the joint density $f(x_1, x_2)= x_1x_2$, $0<x_1<1$, $0<x_2<2$. Find the probability that both random variables will take on values less than 1.**

<u>Soln.</u>

$P(x_1\le1, x_2\le1) = \int_0^1 \int_0^1 x_1x_2\, dx_2dx_1$.

$= \int_0^1 x_1(x_2^2/2)\Big|_0^1\, dx_1$.

$= (1/2) (x_1^2 / 2)\Big|_0^1$

$=1/4$

5. The joint probability mass function of (X,Y) is given by P(x,y) = k (2x + 3y) x = 0,1,2 y = 1,2,3. Find the marginal probability distribution of X

X \ Y	1	2	3
0	3k	6k	9k
1	5k	8k	11k
2	7k	10k	13k

$\sum \sum P(x,y) = 1 \Rightarrow 72\,k = 1 \Rightarrow k = 1/72$

Marginal distribution of X :

X	0	1	2
P(X = x)	18/72	24/72	30/72

6. If f(x,y) = k (1 – x – y), 0 < x,y < ½ . Find K.

$$K \int_0^{1/2}\int_0^{1/2} (1 - x - y)\, dx\, dy = 1 \Rightarrow K \int_0^{1/2} ((1 - y)/2 - 1/8)\, dy = 1 \Rightarrow K \left(3y/8 - y^2/4 \right)_0^{1/2} \Rightarrow k/8 = 1$$

$\Rightarrow K = 8$

7. If X and Y are independent random variables with variances 2 and 3. Find Var(3X + 4Y)

$Var(3X + 4Y) = 3^2\, Var(x) + 4^2\, Var (y) = 9 * 2 + 16 * 3 = 66$

8. If X and Y are independent random variables, find covariance between X + Y and X – Y.

$Cov (X + Y, X - Y) = E [(X + Y) - E (X + Y)) [(X - Y) - E (X - Y)]$

$= E [(X - E(X) + Y - E (Y)) (X - E (Y) - (Y - E (Y))]$

$= E [(X - E(X))^2 - (Y - E (Y))^2$

$= Var\, X - Var\, Y$

9. If X and Y have joint probability density function f(x,y) = x + y 0 < x,y < 1, Check whether X and Y are independent

Marginal density function of X is $f_X(x) = \int_0^1 f(x,y)\, dy = \int_0^1 (x + y)\, dy = x + ½ , 0 < x < 1$

Marginal density function of Y is $f_Y(y) = \int_0^1 f(x,y)\, dx = \int_0^1 (x + y)\, dy = y + ½ , 0 < y < 1$

$f_X(x) . f_Y(y) = (x + ½) . (y + ½) \neq f(x,y)$ X and Y are not independent.

10.Let X and Y be random variables with joint density function f (x, y) = 4xy 0 < x,y < 1. Find E(xy).

$$E(xy) = \int_0^1\int_0^1 xy\, f(x,y)\, dx\, dy = \int_0^1\int_0^1 xy\, 4xy\, dx\, dy = 4\int_0^1\int_0^1 x^2 y^2\, dx\, dy = 4/9$$

11.The joint pdf of two random variables X and Y is given by f(x,y) = 1/8 x (x-y), 0 < x < 2, -x < y < x . Find f (y/x).

$$f_X(x) = \int_{-x}^{x} 1/8\, x\,(x - y)\, dy = x^3/4 \;, 0 < x < 2$$

$$f(y/x) = f(x,y) / f_X(x) = 1/8\, x\,(x-y) / x^3/4 = (x-y) / 2x^2 \;, -x < y < x$$

12. The joint pdf of (X,Y) is $f(x,y) = 6e^{-2x-3y}$, x ≥ 0, y ≥ 0. find the conditional density of Y given X.

Soln.

Marginal of X is $f(x) = \int 6e^{-2x-3y} dy$

$= 2e^{-2x}, x > 0.$

Conditional of Y given X:

$f(y/x) = f(x,y) / f(x)$

$= 6e^{-2x} e^{-3y} / xe^{-2x}.$

$= 3 e^{-3y}, y \geq 0.$

13. State the basic properties of joint distribution of (X,Y) when X and Y are random variable.

Soln.

$F(x,y) = P(X \leq x, Y \leq y)$

$F(\infty, \infty) = 1.$

$F(\infty, y)$ = Marginal distribution of Y. $F(x,\infty)$ = Marginal distribution of X. $F(-\infty, y) = 0$, $F(x, -\infty) = 0$.

14. If 2 random variables have the joint density $f(x_1, x_2) = x_1 x$, $0 < x_1 < 1$, $0 < x_2 < 2$. Find the probability that both random variables will take on values less than 1.

Soln.

$$P[x_1 \leq 1, x_2 \leq 1] = \int_0^1 \int_0^1 x_1 x_2 \, dx_2 \, dx_1$$

$$= \int x_1 (x_2 / 2) \, dx_1 = 1 / 4.$$

15. The conditional p.d.f. of X and Y = y is given by f(x / y) = [(x+y) / (1+y)] e^{-x}, 0< x < ∞, 0 < y < ∞. Find P(x < 1 /Y=2).

Soln.

When y=2, f(x / y=2) =[(x+2) /3] e^{-x}

$$P(X<1 / Y=2) = \int_0^1 [(x+2)/3]\, e^{-x}$$

$$= (1/3)\int_0^1 x\, e^{-x}\, dx + (2/3)\int_0^1 e^{-x}\, dx$$

$$= 1 - (4/3)\, e^{-1}$$

16. If X and Y are independent random variables find covariance between X+Y and X – Y.

Soln.

Cor(X+Y, X-Y) =E [{(X+Y)-E(X+Y)} {(X-Y) – E(X-Y)]

= E[{ X- E(X) +Y – E(Y)} { (X-E(Y)) – (Y-E(Y))}]

= E[$\{X-E(X)\}^2 - \{Y - E(Y)\}^2$

= Var X – Var Y.

17. If X and Y are independent random variable with variances 2 and 3, find Var (3X +4Y).

Soln.

Var(3X+ 4Y) = 3^2 Var(X) + 4^2 Var(Y)

= 9 X 2 + 16 X 3

= 66.

18. If the joint pdf of (X,Y) is given by f(x,y) = $e^{-(x+y)}$, x ≥ 0, y ≥ 0 Find E (XY)

$$E(XY) = \int_0^\infty\int_0^\infty xye^{-(x+y)}dxdy = \int_0^\infty xe^{-x}dx\int_0^\infty ye^{-y}dy = 1$$

19.**The regression lines between two ransom variables X and Y is given by 3x + 4y =10 and 3x + 4y = 12. Find the correlation between two regression lines**.

3x + 4y =10 ⇒ $b_{yx} = -\frac{3}{4}$. 3x + 4y = 12 ⇒ $b_{xy} = -\frac{1}{3}$

$r^2 = -\frac{3}{4} \cdot -\frac{1}{3} = \frac{1}{4} \Rightarrow r = \frac{1}{2}$

20. **Distinguish between correlation and regression**

By correlation we mean the casual relationship between two or more variables. By regression we mean the average relationship between two or more variables.

21. **Why there are two regression lines?**

Regression lines express the linear relationship between two variable X and Y. Since any of them is taken as independent variable, we have two regression lines.

22. **State central limit theorem?**

If $X_1, X_2 \ldots X_n$ be a sequence of independent identically distributed R.Vs with $E(X_i) = \mu$ and $var(X_i) = \sigma^2$ for o = 1,2… and if $S_n = X_1 + X_2 + \ldots X_n$, then under certain general conditions, S_n follows a normal distribution with mean $n\mu$ and variance $n\sigma^2$ as n tends to ∞

23.**If Y_1 and Y_2 are two independent random variables, then covariance (Y_1, Y_2) = 0. IS the converse of the above statement true? Justify your answer.**

The converse not true. Consider $X \sim N(0,1)$ and $Y = X^2$ since $X \sim N(0,1)$. E (X) = 0; E (X^3) = E (XY) = 0 since all odd moments vanish. Cov (x, y) = 0 but X and Y are not independent.

24. **Find the value of k, if f(x,y) = k (1- x) (1- y) for 0 < x,y < 1 is to be a joint density function**.

We know $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1 \Rightarrow \int_0^1\int_0^1 k(1-x)(1-y)dxdy = 1$

$$k\int_0^1\left(x-\frac{x^2}{2}-xy+\frac{x^2y}{2}\right)_0^1 dy = 1 \Rightarrow k\int_0^1\left(\frac{1}{2}-y+\frac{y}{2}\right)dy = 1 \Rightarrow k\left(\frac{y}{2}-\frac{y^2}{2}+\frac{y^2}{4}\right)_0^1 = 1$$

$$k\left(\frac{1}{4}\right) = 1 \Rightarrow k = 4$$

25.If X has an exponential distribution with parameter α find the pdf of Y = log X

$$f_Y(y) = f_X(x)\left|\frac{\mathrm{dx}}{\mathrm{dy}}\right| = e^y \alpha e^{-\alpha e^{-y}}, -\infty < y < \infty$$

Part- B Questions

1. If f (x, y) = x+y , 0< x <1, 0< y < 1
   0 , Otherwise
   Compute the correlation cp- efficient between X and Y.

2. The joint p.d.f of a two dimensional randm variable (X, Y) is given by f(x, y) = (8 /9) xy, $1 \le x \le y \le 2$ find the marginal density unctions of X and Y. Find also the conditional density function of Y / X =x, and X / Y = y.

3. The joint probability density function of X and Y is given by f(x, y) = (x + y) /3, $0 \le x \le 1$ & 0<y <2 obtain the egression of Y on X and of X on Y.

4. If the joint p.d.f. of two random variable is given by $f(x_1, x_2) = 6\, e^{-2x_1 - 3x_2}$, $x_1 > 0$, $x_2 > 0$. Find the probability that the first random variable will take on a value between 1 and 2 and the second random variable will take on avalue between 2 and 3. \also find the probability that the first random variable will take on a value less than 2 and the second random variable will take on a value greater than 2.

5. If two random variable have hoing p.d.f. $f(x_1, x_2) = (2/3)(x_1 + 2x_2)$ $0 < x_1 < 1$, $0 < x_2 < 1$

6. Find the value of k, if f(x,y) = k xy for 0 < x,y < 1 is to be a joint density function. Find P(X + Y < 1) . Are X and Y independent.

7. If two random variable has joint p.d.f. $f(x, y) = (6/5)(x + y^2)$, 0 < x < 1 , 0< y <1.Find P(0.2 < X < 0.5) and P(0.4 < Y < 0.6)

8. Two random variable X and Y have p.d.f $f(x, y) = x^2 + (xy/3)$, $0 \le x \le 1$, $0 \le y \le 2$. Prove that X and Y are not independent. Find the conditional density function

9. X and Y are 2 random variable joint p.d.f. $f(x, y) = 4xy\, e^{-(x^2+y^2)}$, $x, y \ge 0$, find the p. d. f. of $\sqrt{x^2+y^2}$.

10. Two random variable X and Y have joint f(x y) 2 – x –y, 0< x <1, 0< y < 1. Find the Marginal probability density function of X and Y. Also find the conditional density unction and covariance between X and Y.

11. Let X and Y be two random variables each taking three values –1, 0 and 1 and having the joint p.d.f.

Y	X		
	-1	0	1
-1	0	0.1	0.1
0	0.2	0.2	0.2
1	0	0.1	0.1

Prove that X and Y have different expections. Also Prove that X and Y are uncorrelated and find Var X and Var Y

12. 20 dice are thrown. Find the approximate probability tat the sum obtained is between 65 and 75 using central limit theorem.

13. Examine whether the variables X and Y are independent whose joint density is $f(x, y) = x e^{-xy-x}$, $0 < x, y < \infty$

.

14. Let X and Y be independent standard normal random variables. Find the pdf of $z = X / Y$.

15. Let X and Y be independent uniform random variables over (0,1) . Find the PDF of $Z = X + Y$

16. If x and y are two dimensional random variables having joint density function $f(x,y)=\begin{cases} \frac{1}{8}(6-x-y) & 0<x<2, 2<y<4 \\ 0 & o.w \end{cases}$ find the marginal density function and

P(x<1∩y<3)   P(x<1/y<3) & P(x+y<3).

UNIT-III MARKOV PROCESS AND MARKOV CHAIN

1. Define a Random process and give an example of a random process.

A random process is a collection of random variables $\{X(s,t)\}$ that are function of time $s \in S$ and $t \in T$.

Example:

$$x(t) = A\cos(\omega t + \theta)$$

Where θ = uniform distributed in (0, 2π)

A, ω = constants.

2. State the four types of a stochastical processes.

Discrete time, discrete state

Discrete time, continuous state

Continuous time, discrete state

Continuous time, continuous state

3. If $\{X(s,t)\}$ is a random process, what is tha nature of $X(s,t)$ when (i) S is fixed (ii) t is fixed.

(i) When S is fixed, $X(s,t)$ is a time function.

(ii) When t is fixed, $X(s,t)$ is a random variable

4. What is discrete Random sequence? Give an example.

If both state space S and parameter set T are discrete, then the random process is called a discrete random sequence. If X_n represents the outcome of the n^th^ toss of a fair dies, then $\{X_n,\ n \geq 1\}$ is discrete random sequence.

5. Define discrete random process. Give example

If T is continuous and S is discrete, the random process is called a discrete random process.

Example:

If X(t) represents the number of telephone calls received in the interval (o, t), then $\{X(t)\}$ is a discrete random process.

6. What is a continuous random process. Give example.

If both S and T are continuous, the random process is called a continuous random process.

Example: If X(t) represents the maximum temperature at a place in the interval (o, t), $\{X(t)\}$ is a continuous random process.

7. **What do you mean by the Mean and Variance of a random process?**

If $X(t)$ is a representative members function of the random process $\{X(t)\}$, $E[X(t)]$ and $Var[X(t)]$ are called the Mean and Variance of the process

8. **When are two random processes said to be orthogonal?**

Two random processes $\{X(t)\}$ $\{Y(t)\}$ are said to be orthogonal. If $E[X(t_1)Y(t_2)] = 0$

9. **Define the Auto correlation and Auto ovariance of Random process** $\{X(t)\}$

Autocorrelation Function

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$

Auto covariance

$$C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - E[X(t_1)]E[X(t_2)]$$

10. **Define the Cross Correlation of two random process.**

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$

11. **Define Cross Covariance of two random processes**

$$R_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - E[X(t_1)]E[Y(t_2)]$$

12. **Define a Stationary Process**

If certain probability distribution or averages do not depend on time t, then the random process $\{X(t)\}$ is called stationary process.

13. **Define Strict Sense Stationary Process.**

A random process $\{X(t)\}$ is a SSS process. If the joint distribution of $X(t_1), X(t_2), ...X(t_n)$ is the same as that of $X(t_1+h), X(t_2+h), ...X(t_n+h)$ for all $t_1, t_2, ...t_n$ and h > 0 and for all n ≥ 1

Example: Bernoulli's Process.

14. Define Second order Stationary process

A random process $\left\{X(t)\right\}$ is said to be second order SSS, if $f(x_1,x_2,t_1,t_2)=f(x_1,x_2,t_1+h,t_2+h)$ where $f(x_1,x_2,t_1,t_2)$ is the joint p.d.f $\{X(t_1),X(t_2)\}$

15. Define Wide Sense Stationary Process

A random process $\left\{X(t)\right\}$ is called wide sense stationary. If $\mathrm{E}[X(t)]$ is a constant and $\mathrm{E}[X(t)X(t+\tau)]=R_{XX}(z)$

(i.e) A.C.F. is a function of Z only.

16. Define evolutionary process and give example.

A random process $\left\{X(t)\right\}$ that is not stationary in any sense is called an evolutionary process.

Example: Poisson process.

17. When are the processes $\left\{X(t)\right\}$ and $\left\{Y(t)\right\}$ said to be jointly stationary in the Wide Sense?

Two random process $\left\{X(t)\right\}$ and $\left\{Y(t)\right\}$ said to be jointly stationary in the Wide sense if each process is individually a WSS process and (t_1,t_2) is a function of (t_1,t_2) only.

18. State any four properties of Auto Correlation Function

(1) $R_{XX}(-Z)=R_{XX}(Z)$

(2) $|R(Z)|\le R(0)$

(3) $R(Z)$ is continuous for all Z

(4) If $R(Z)$ is A.C.F. of a stationary R.P. $\left\{X(t)\right\}$ with no periodic components, then

$$\mu_X{}^2=\lim_{z\to\infty}R(Z)$$

19. Give that the Auto correlation function for a stationary ergodic process with no periodic components is $R(Z)=25+\dfrac{4}{1+6Z^2}$. Find the Mean and variance of the Process $\left\{X(t)\right\}$

Soln:

$$\mu_X{}^2 = \lim_{z\to\infty} R(Z) = 25$$
$$= 5$$

$$\mathrm{E}[X^2(t)] = R_{XX}(0) = 25 + 4$$
$$= 29$$

$$Var[X(t)] = E[X^2(t)] - \mathrm{E}^2[X(t)]$$
$$= 29 - 25$$
$$= 4$$

20. Find the Mean and Variance of the stationary process $\{X(t)\}$ whose A.C.F. $R(Z) = \dfrac{25Z^2 + 36}{6.25Z^2 + 4}$

Soln:

$$\mu_X{}^2 = \lim_{z\to\infty} R(Z)$$

$$= \lim_{T\to\infty} \frac{25 + 36/Z^2}{6.25 + 4/Z^2}$$

=4

$$\mu_X = 2$$

$$E[X^2(t)] = R_{XX}(0)$$
$$= 9$$

$$V[X(t)] = 9.4$$
$$= 5$$

21. A stationary random process [X(t)] with mean 4 has ACF R(Z) = 16 + 9 $e^{-|z|}$. Find the standard deviation of the process

$$\mathrm{E}[X^2(t)] = R_{XX}(o)$$
$$= 16 + 9 = 25$$
$$V[X(t)] = 25 - 16 = 9$$

Standard deviation =3

22. Define the Cross- Correlation function and state any two of its properties.
Soln:

$$R_{XY}(t_1,t_2) = \mathrm{E}[X(t_1)X(t_2)]$$
$$(1) R_{XY}(Z) = R_{XX}(-Z)$$
$$(2) |R_{XY}(\mathrm{Z})| \le \sqrt{R_{XX}(0)R_{YY}(0)}$$
$$\le \frac{1}{2}[R_{XX}(0)+R_{YY}(0)]$$

23. When is a random process said to be ergodic? Give example?

A random process $\{X(t)\}$ is ergodic if its ensembled averages equal to appropriate time averages.

Example: $X(t) = A\cos(\omega t + \theta)$ $\theta =$ uniformly distributed in (0, 2π) is Mean ergodic.

24. Distinguish between stationary and ergodicity.

Stationary of a random process is the property of the process by which certain probability distributions or averages do not depend on time t.

Ergodocity of a random process is the property by which almost every member of the process exhibits the same statistical behavior as the whole process.

25. Check for the stationary of the random process $X(t) = A\cos(\omega t + \theta)$ if A and ω_0 are constants and θ is a uniformly distributed RV in (0, 2π)

$$\mathrm{E}[X(t)] = 0 \text{ and } R_{XX}(t, t+z) = \frac{A^2}{2}\cos\omega_0 z$$

$\Rightarrow [X(t)]$ is wide sense stationary.

26. Give that R.P. $\{X(t)\} = A\cos\omega_0 t + B\sin\omega_0 t$ where ω_0 is constant and A and B are uncorrelated zero mean random variables having different densities but the same variance. Is $\{X(t)\}$ wide sense stationary?

$$\mathrm{E}[X(t)] = 0$$
$$\mathrm{E}[X^2(t)] = \sigma^2$$
$$and\ \mathrm{R}_{XX}(t_1,t_2) = \sigma^2\cos\omega_0(t_1 - t_2)$$

$X(t)$ is Wide sense stationary.

27. Define Markov Process.

If the future behaviour of a Process depends only the present state but not on the past, the process is a Markov process.

28. Define Markov chain

A discrete parameter Markov process is called a Markov chain (or)If,forall

$$P[X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, ...X_0 = a_0] = P[X_n = a_n / X_{n-1} = a_{n-1}]$$

then the process $\{X_n\}$, n= 0, 1, 2, ……. Is called Markov Chain $(a_1, a_2, ...a_n, ...)$ called the states of the Markov Chain.

29. Define A Markov Process.

If for $t_1 < t_2 < ...... < t_n < t,$

$$P[X(t) \le x / X(t_1) = x_1, X(t_2) = x_2, ...X(t_n) = x_n] = P[X(t) \le x / X(t_n) = x_n]$$

Then the process $\{X(t)\}$ is called Markov Process.

30. What is a Stochastic matrix? When is it said to be regular.

If $P_{ij} \ge 0$ and $\sum P_{ij} = 1$ for all I, then the t.p.m. $P = (P_{ij})$ of a Markovchain is a stochastic matrix P. A stochastic matrix P is said to be regular matrix, if all the entries of P^m (for some +ve integer m) are positive.

31. Define irreducible Markov chain. Also state Chapman – Kolmogrov Theorem.

If $P_{ij}^{(n)} > 0$ for some n and for all i and j, then every state can be reached from every other state. When this condition is satisfied, the Markov Chain is said to be irreducible

Theorem:

If P is the t.p.m. of a homogeneous Markov chain, then n step t.p.m. $P^{(n)}$ is equal to p^n.

32. What is a Markov chain? When can you say that a Markov chain is homogeneous?

A discrete state stochastic process $\{X(t) / t \in T\}$ is a Markov Chain.

If
$[X(t) \le x / X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, ...X(t_0) = x_0] = P[X(t) \le x / X(t_n) = x_n]$
If it has the property of invariance with respect to the time origin.

(i.e) $P[X(t) \le x / X(t_n) = x_n] = P[X(t - t_n) \le x / X(0) = x_n]$,
then the Markov Chain is homogeneous.

33. Consider the random process $\{X(t)\} = \cos(\omega_0 t + \theta)$ where θ is uniformly distributed in the interval (-π, π). Check whether X(t) is stationary or not?

$$\begin{aligned} \mathrm{E}[X(t)] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\omega_0 t + \theta) d\theta \\ &= \frac{1}{2\pi}[\sin(\omega_0 t + \pi) - \sin(\omega_0 t - \pi)] \\ &= \frac{1}{2\pi}[-\sin \omega_0 t + \sin \omega_0 t] \\ &= 0 \end{aligned}$$

$$\begin{aligned} \mathrm{E}[X^2(t)] &= \frac{1}{4\pi}\left[\frac{\theta - 2\sin(\omega_0 t + \theta)}{2}\right] \\ &= \frac{1}{2} \end{aligned}$$

The process is stationary as the first and second moments are independent of time.

34. The one- step t.p.m. of a Markov Chain with states 0 and 1 is given as $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Draw the transition diagram. Is it irreducible Markov chain?

Yes, it is irreducible, because each state can be reached from any other state.

35. Three boys A, B, C are throwing a ball to each other. A always throw the ball to B and B always throws the ball to C but C is just as likely to throw the ball B as to A. Find the transition Matrix and show that the process is Markovian.

TPM of $\{X(t)\}$ is given by P

$$P = X_{n-1}\overset{X_n}{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix}}$$

Since X_n depends only on X_{n-1} but not on states of $X_{n-2}, X_{n-3}, ...\{X_n\}$ is a Markov process.

36. What do you mean by an absorbing Markov chain? Give an example.

A state i of a Markov chain is said to be an absorbing state if P_{ij} =1.(i.e.) if it is impossible to leave it. A markov chain is said to be absorbing if it has atleast one absorbing state.

The TPM of an absorbing Markov Chain is

$$tpm = \begin{matrix} \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix}\begin{matrix} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

37. Prove that the matrix $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix}$ is the t.p.m. of an irreducible Markov chain.

$$P^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

$P_{11}{}^{(3)} > 0, P_{13}{}^{(2)} > 0, P_{21}{}^{(2)} > 0, P_{22}{}^{(2)} > 0, P_{33}{}^{(2)} > 0$ and for all other $P_{ij}{}^{(1)} > 0$, the chain is irreducible.

38. State the postulates of a poisson process.

Let X(t) = number of times an event A say occurred upto time 't' so that the sequence $\{X(t)\}$, t ≥ 0 forms a Poisson process with parameter λ.

(i) Events occurring in non- overlapping intervals are independent of each other
(ii) P[X (t) =1 for t in (x , x + h) = λh + 0(h)]
(iii) P[X(t) = 0 for t in (x , x + h) = 1- λh + 0(h)]
(iv) P[X(t) = 2 or more for t in (x , x + h)] = 0]
(v)

39. State any two properties of Poisson Process.

(i) The Poisson process is a Markov Process.
(ii) Sum of two independent Poisson processes is a Poisson process.
(iii) The difference of two independent Poisson processes is not a Poisson process.
(iv)

40. What will be the super position of n independent Poisson processes with respective averages rates $\lambda_1, \lambda_2, ...\lambda_n$?

The super position of n independent Poisson processes with averages rates $\lambda_1, \lambda_2, ...\lambda_n$ is another Poisson process with averages rate $\lambda_1 + \lambda_2 + ... + \lambda_n$

41. If the customers arrive at a counter in accordance with a Poisson process with a mean rate of 2 per minute. Find the probability that the interval between two consecutive arrivals is more than one minute?

The interval T between 2 consecutive arrivals follows an exponential distribution with parameter λ = 2, $P[T > 1] = \int_1^\infty 2e^{-2t} = e^{-2} = 0.135$

42. **Let X(t) be a Poisson process with rate λ. Find correlation function of X (t).**

43. For Z > 0,

$$\begin{aligned} E[X(t).X(t+z)] &= E\{X(t)[X(t+z) - X(t) + X(t)]\} \\ &= E[X(t)]E[X(t+z) - X(t)] + E[X^2(t)] \\ &= \lambda t(\lambda z) + \lambda^2 t^2 + \lambda t \\ &= \lambda^2 tz + \lambda^2 t^2 + \lambda t \end{aligned}$$

44. Show that the Poisson process is not Covariance stationary?

$$P[X(t)] = \frac{e^{-\lambda t}(\lambda t)^r}{r!}, r = 0,1,2,....$$

E[X(t)] = λt $\neq$ a constant.   The process is not covariance stationary.

45. A bank receives on the average λ = 6 bad checks per day, what are the probabilities that it will receive (i) 4 bad checks on any given day, (ii) 10 bad checks over any two consecutive days?

$$P[X(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$= \frac{e^{-6t}(6t)^n}{n!}, n = 0,1,2,....$$

$$(i) P[X(1) = 4] = \frac{e^{-6}(6)^4}{4!} = 0.1338$$

$$(ii) P[X(2) = 10] = \frac{e^{-12}(12)^{10}}{10!} = 0.1048$$

46. Customers arrive a large store randomly at an average rate of 240 per hour. Shat is the probability that during a two- minute interval no one will arrive?

$$P[X(t) = n] = \frac{e^{-4t}(4t)^n}{n!}, n = 0,1,2,...$$

$$\text{since } \lambda = \frac{240}{60} = 4$$

$$P[X(2) = 0] = e^{-8} = 0.0003$$

47. The number of arrivals at the Reginal computer center at express service counter between 12 noon and 3 pm has a poisson distribution with a Mean of 1.2 perminute. Find the probability of no arrivals during a given 1- minute interval.

$$P[X(t) = n] = \frac{e^{(1.2)t}\{1.2t\}^n}{n!}, n = 0,1,2,...$$

$$P[X(1) = 0] = e^{-1.2} = 0.3012$$

48. Define Gaussian or normal process (or) when is a random process is said to be normal?

A real valued R.P. $\{X(t)\}$ is called a Gaussian process or normal process if the random variables $X(t_1), X(t_2), ...X(t_n)$ are jointly normal for any n and for any set $t_i\,'s$

49. State the properties of Gaussian process.

(i) If a Gaussian process is wide sense stationary, it is also strict sense stationary

(ii) If the member functions of a Gaussian process are uncorrelated, the they are independent.

(iii) If the input $\{X(t)\}$ of a linear system is a Gaussian process, the output will also be a Gaussian process.

(iv)

50. Name a few random processes that are defined interms of stationary Gaussian random process.

(i) Square law detector process

(ii) Full- Wave linear detector process

(iii) Half- Wave linear detector process

(iv) Hard limiter process.

51. If $\{X(t)\}$ is a Gaussian process with $\mu(t) = 10$ and

$C(t_1, t_2) = 16e^{-|t_1 - t_2|}$ **Find** $P\{X(10) \le 8\}$

$\mu[X(10)] = 10$ and Var[X(10)]=C(10,10)=16

$$P[X(10) \le 8] = P\left[\frac{X(10) - 10}{4} \le -0.5\right]$$

$$= P[z \le -0.5]$$

$$= 0.5 - P[z \le 0.5]$$

$$= 0.5 - 0.1915$$

$$= 0.3085$$

52. Given a normal process $\{X(t)\}$ with zero mean and $R_{XX}(z) = 4e^{-2\,|z|}$. Find Var[X(t)].

$$Var[X(t)] = R_{XX}(z) - \mathrm{E}^2[X(t)]$$

$$= 4e^{-2|z|}$$

53. If X(t) is a normal process with $C(t_1, t_2) = 4e^{-0.5|z|}$, what is the variance of X(5).

$$Var[X(t)] = C(5,5) = 4e^{-0.5(0)}$$
$$= 4, \qquad \text{since } z = t_1 - t_2$$

54. If $\{X(t)\}$ is a normal process with $\mu(t) = 3$ and $C(t_1 - t_2) = 4e^{-0.2|t_1 - t_2|}$ find the variance of X(8) – X(5)

$$Var[X(8) - X(5)] = \text{var}[X(8) + X(5)] - 2\,\text{cov}[X(8), X(5)]$$
$$= 4 + 4 - 2 \text{ X}4 \text{ X e}^{-0.6}$$
$$= 8[1 - \text{e}^{-0.6}]$$
$$= 3.6095$$

PART B QUESTIONS

1. The one step transition probability matrix of a Markov chain { X_n ; n = 0,1..} having state space S = { 1,2,3} is $P = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$ and the initial distribution is π = (0.7 0.2 0.1). Find (1) P ($X_2 = 3 / X_0 = 1$) (2) P ($X_2 = 3$) (3) P ($X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 1$)

2. The process { X(t) } whose proabability distribution is given by

$$\text{P [X(t) = n] = } \frac{(at)^{n-1}}{(1+at)^{n+1}}, n = 1, 2...$$
$$= \frac{(at)}{1+at}, n = 0$$

Show that it is not stationary

3. A raining process is considered as a two state Markov chain. If it rains, it is considered to be in state 0 and if it does not rain, the chain is in state 1. the transitioin probability of the markov chain is defined as $P = \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix}$. Find the probability of the Markov chain is defined as today assuming that it is raining today. Find also the unconditional probability that it will rain after three days with the initial probabilities of state 0 and state 1 as 0.4 and 0.6 respectively.

4. Let X(t) be a Poisson process with arrival rate λ. Find E {(X (t) – X (s)^2 } for t > s.

5. Let { X_n ; n = 1,2..} be a Markov chain on the space S = { 1,2,3} with on step transition probability matrix $P = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{pmatrix}$ (1) Sketch transition diagram (2) Is the chain irreducible? Explain. (3) Is the chain Ergodic? Explain
6. Consider a random process X(t) defined by X(t) = U cost + (V+1) sint, where U and V are independent random variables for which E (U) = E(V) = 0 ; E (U^2) = E (V^2) = 1 (1) Find the auto covariance function of X (t) (2) IS X (t) wide sense stationary? Explain your answer.
7. Discuss the pure birth process and hence obtain its probabilities, mean and variance.
8. At the receiver of an AM radio, the received signal contains a cosine carrier signal at the carrier frequency ω with a random phase θ that is uniform distributed over (0,2π). The received carrier signal is X (t) = A cos(ωt + θ). Show that the process is second order stationary
9. Assume that a computer system is in any one of the three states busy, idle and under repair respectively denoted by 0,1,2. observing its state at 2 pm each day, we get the transition probability matrix as $P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.6 & 0 & 0.4 \end{pmatrix}$. Find out the 3rd step transition probability matrix. Determine the limiting probabilities.
10. Given a random variable Ω with density f (ω) and another random variable φ uniformly distributed in (-π, π) and independent of Ω and X (t) = a cos (Ωt + φ), Prove that { X (t)} is a WSS Process.
11. A man either drives a car or catches a train to go to office each day. He never goes 2 days in a row by train but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of the week, the man tossed a fair die and drove to work iff a 6 appeared. Find (1) the probability that he takes a train on the 3rd day. (2) the probability that he drives to work in the long run.
12. Show that the process X (t) = A cosλt + B sin λt (where A and B are random variables) is wide sense stationary, if (1) E (A) = E(B) = 0 (2) E(A^2) = E (B^2) and E(AB) = 0
13. Find probability distribution of Poisson process.
14. Prove that sum of two Poisson processes is again a Poisson process.
15. Draw the state diagram of a birth death process. Write down the balance wquations and obtain expression for the steady state probabilities.
16. Write classifications of random processes with example.

17. A radioactive source emits particles at rate of 5 per minute in accordance with Poisson process. Each particle emitted has a probability 0.6 of being recorded. Find the probability that 10 particles are recorded in 4 – min period.

18. Patients arrive randomly and independently at a doctor's consulting room from 8 A.M. at an average rate of one in 5 min. The waiting room can hold 12 persons. What is the probability that the room will be full when the doctor arrives at 9 A.M.?

19. Suppose that the customers arrive at a counter independently from 2 different sources. Arrivals occur in accordance with a Poisson process with mean rate of 6 per hour from the first source and 4 per hour from the second source. Find the mean interval between any 2 successives arrivals.
20. Define Autocorrelation and Autocovariance of the Random process.
Define Correlation coefficient, Cross-correlation,crosscovariance of Random process. Define Random walk. Define a semirandom telegraph signal process and random telegraph signal process . Are they stationary? (Pg -353)
21. State and prove the additive property of Poisson process.
Prove that the difference of 2 independent Poisson Process is not a Poisson Process.
22. A man either drives a car or catches a train to go to office each day. He never goes 2 days in a row by train but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of the week, the man tossed a fair dice and drove to work if and only if a 6 appeared. Find (a) the probability that he takes a train on the third day and (b) the probability he drives to work in the

23. A gambler has Rs. 2/- . He bets Rs.1 at a time and wins Rs.1 with probability .. He stops playing if he loses Rs.2 or wins Rs.4. (a)what is the tpm of the related Markov chain? (b) What is the probability that he has lost his money at the end of 5 plays? (c) What is the probability that the same game lasts more than 7 plays?

24. There are 2 white marbles in urn A and 3 red marbles in urn B. At each step of the process, a marble is selected from each urn and the 2 marbles selected are interchanged. Let the state aI of the system be the number of red marbles in A after i changes. What is the probability that there are 2 red marbles in urn A?

25. Three boys A,B and C are throwing a ball to each other. A always throws the ball to B and B always throws the ball to C, but C is just as likely to throw the ball to B as to A. Show that the process is Markovian. Find the transition matrix and classify the states.

UNIT IV: QUEUEING THEORY

1. What are the characteristics of a queuing system?

(1) The input (or arrival) pattern – It describes the manner in which the customers arrive and join the queuing system.
(2) The service mechanism (pattern): It describes the mode of service and specifies the number of servers and the arrangement of servers (in parallel, in series etc)
(3) The queue discipline.

2. What is meant by queue discipline?

It specifies the manner in which the customers form the queue or equivalently the manner in which they are selected for service; when a queue had been formed. The most common discipline are (i) FCFS (first come first served) (ii) LCFS (Last come first served) (iii) SIRO (Service in random order)

3.**Define Little's formula.**

(i) $L_s = \lambda W_S$ (ii) $L_q = \lambda W_q$ (iii) $W_S = W_q + 1/\mu$ (iv) $L_s = L_q + \lambda / \mu$

4. **If the people arrive to purchase cinema tickets at the average rate of 6 per minute, it takes an average of 7.5 seconds to purchase a ticket. If a person arrives 2 minutes before the picture starts and if it takes exactly 1.5 minutes to reach the correct seat after purchasing the ticket. Can he expect to be seated for the start of the picture?**

Here $\lambda = 6$, $1 / \mu = 15 / 2 \Rightarrow \mu = 2 / 15$ per sec $\Rightarrow \mu = 2/15 * 60 = 8$

$E(L_q) = 1 / (\mu - \lambda) = 1/(8-6) = ½$ min . He can just seated for the start of the picture

E(Total time) = ½ + 3/2 = 2 min

5. **If the arrival and departure rates in a public telephone booth with a single phone are 1/12 and 1 / 4 respectively, find the probability that the phone is busy**

P(phone is busy) = 1 – P (no customer in the booth) = $1 - P_0 = 1 - (1 - \lambda / \mu) = \lambda / \mu = 1 / 3$

6**.If the inter arrival time and service time in a public telephone booth wilth a single phone follow exponential distributions with means of 10 and 8 min respectively, find the average number of callers in the booth at any time.**

Here $\lambda = 1/10$ and $\mu = 1/8$. $L_s = \lambda / \mu - \lambda = 1/10 * 40 = 4$

7. **If the arrival and departure rates in a M/M/1 queue are ½ per minute and 2/3 per minute respectively, find the average waiting time of a customer in the queue**

$E(W_q) = \lambda / \mu(\mu - \lambda) = ½ /2/3(2/3 – ½) = 4.5$ min

8. **Customers arrive at a railway ticket counter at the rate of 30 / hour. Assuming Poisson arrivals, exponential service time distribution and a single server queue(M/M/1) model, find the average waiting time(before being served) if the average service time is 100 sec.**

Here $\lambda = 30$ / hour and $1/\mu = 100$ sec = 1/36 hour $\mu = 36$/hour

Average waiting time in the queue $E(W_q) = \lambda / \mu(\mu - \lambda) = 30/36*6 = 5/36h = 8.33$ min

9.**What is the probability that a customer has to wait more than 15 min to ger his service completed in (M/M/1) : (∞ / FIFO) queue system if λ = 6 per hour and μ = 10 per hour?**

Probability that the waiting time of a customer in the system exceeds time t = $e^{-(\mu-\lambda)t}$. Here $\lambda = 6$ $\mu = 10$ t = 15 min = ¼ hour . Required probability = $e^{-1} = 1/e$.

10. **What is the probability that a customer has to wait more than 15 minutes to get his service completed in M/M/1 queuing system, if λ = 6 per hour μ = 10 per hour?**

Probaility that the waiting time in the system exceeds is

$$P(W_S > t) = \int_t^{\infty} (\mu-\lambda)e^{-(\mu-\lambda)\omega} d\omega = e^{-(\mu-\lambda)t} = e^{-(10-6)¼} = e^{-1}$$

11. **What is the probability that an arrival to an infinite capacity 3 server Poisson queuing system with $\frac{\lambda}{\mu} = 2$ and $p_0 = \frac{2}{9}$ enters the service without waiting?**

P [without waiting] = P [N < 3] = $P_0 + P_1 + P_2$

$P_n = \frac{1}{n!}(\frac{\lambda}{\mu})^n P_0$ when n ≤ C. Here C = 3 . P [N < 3] = $\frac{1}{9}+\frac{2}{9}+\frac{1}{2}*\frac{4}{9}=\frac{5}{9}$

12. **Consider an M/M/1 queuing system. If λ = 6 and μ = 8, find the probability of at least 10 customers in the system**

Probability that the number of customers in the system exceeds k ,

$$P(n>k)=(\frac{\lambda}{\mu})^{k+1} \Rightarrow P(n>10)=\sum_{10}^{\infty} P_n = \sum_{10}^{\infty}(1-\frac{1}{6})(\frac{6}{8})^{10} = (\frac{6}{10})^{10} = (\frac{3}{10})^{10}$$

13.**Consider an M/M/C queuing system. Find the probability that an arriving customer is forced to join the queue.**

P [a customer is forced to join queue] = $\sum_{c}^{\infty} P_n$

$$= P_0 \frac{(C\rho)^c}{C!(1-\rho)} = \frac{\frac{(C\rho)^c}{C!(1-\rho)}}{\sum_{n=0}^{c-1} \frac{(C\rho)^n}{n!} + \frac{(C\rho)^n}{C!(1-\rho)}}$$

14. **Write down Pollaczek – Khinchine formule**

(i) Average number of customers in the system

$$\frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} + \rho$$

(ii) Average queue length

$$\frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} where \rho = \lambda E(T) \sigma^2 = V(T)$$

15. **Briefly describe the M/G/1 Queuing system**

Poisson arrival / General Service/ single server queuing system

16.**Customers arrive at a one man barber shop according to a Poisson process with mwan inter arrival time of 12 min, Customers spend an average of 10 min in the barber's chair. What is the expected number of customers in the barber shop and in the queue? How much time can a customer expected to spend in the barber's shop?**

$$E(N_S) = \frac{\lambda}{\mu-\lambda} = \frac{\frac{1}{2}}{\frac{1}{10} - \frac{1}{12}} = 5 customers$$

$$E(N_q) = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\frac{1}{144}}{\frac{1}{10}(\frac{1}{10} - \frac{1}{12})} = 4.17 customers$$

$$E(\omega) = \frac{1}{\mu-\lambda} = \frac{1}{\frac{1}{10} - \frac{1}{12}} = 60 \min$$

17.**A duplicating machine maintained for office use is operated bu office assistant. The time to complete each job varies according to an exponential distribution with mean 6 min. Assume a Poisson input with an average arrival rate of 5 jobs per hour. If an hour day is used as a base, determine**

(i) The percentage of idle time of the machine

(ii) The average time a job is in the system

λ = 5/hour μ = 60/6 = 10/hour

(i) P[the machine is idle] = $P_0 = 1 - \lambda/\mu = 1/2 = 50\ \%$

(ii) $E(\omega) = \frac{1}{\mu - \lambda} = \frac{1}{10-5} = \frac{1}{5}\, hours$

18. **Customers arrive at a one man barber shop according to a poisson process with a mean inter arrival time of 12 min. Customers spend an average of 10 minutes in the barber's chair, what is the probability that more than 3 customers in the system?**

$P[N > 3] = P_4 + P_5 + \ldots$

$= 1 - [P_0 + P_1 + P_2 + P_3] = 0.4823$

19.**If there are two servers in an infinite capacity Poisson queue system with λ = 10. μ = 15 /hour , what is the percentage of idle time for each server?**

$$P_0 = \frac{1}{\sum_0^1 \frac{1}{n!}(\frac{2}{3})^n + \frac{1}{2!(1-\frac{1}{3})}(\frac{2}{3})^2} = \frac{1}{2}$$

20.**If λ = 4 per hour and μ = 15 per hour in an (M/M/1) : (4/ FIFO) queuing system, find the probability that there is no customer in the system.**

$$P_0 = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} = \frac{1-\frac{1}{3}}{1-\left(\frac{1}{3}\right)^{4+1}} = \frac{81}{121}$$

21. **Consider an M/M/C queuing system. Find the probability that an arriving customer is forced to join the queue**

P [N ≥ C] = $\sum_{C}^{\infty} P_n = \sum_{C}^{\infty} \frac{1}{C!C^{n-c}} \left(\frac{\lambda}{\mu}\right)^n P_0$

$$= \frac{1}{C!}\left(\frac{\lambda}{\mu}\right)^c P_0 \sum_{c}^{\infty}\left(\frac{\lambda}{C\mu}\right)^{n-c} = \frac{\left(\frac{\lambda}{\mu}\right)^c P_0}{C!\left(1-\frac{\lambda}{C\mu}\right)}$$

22.**For (M/M/C) : (N/FIFO) model, write down the formula for (a) Average number of customers in the queue (b) Average waiting time in the system**

(i) E (W_q) or L_q = $\frac{P_0(\rho C)^C \rho}{C!(1-\rho)^2}\left[1-\rho^{N-C}-(1-\rho)(N-C)\rho^{N-C}\right]$

Where $\rho = \frac{\lambda}{C\mu}$ and $P_0 = \left[\sum_{0}^{C-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{C}^{N}\frac{1}{C^{n-c}}\frac{1}{C!}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1}$

(ii) E (W_S) = L_S / λ' where λ' = λ(1- P_N) or $\mu\left[C-\sum_{0}^{C-1}(C-n)P_n\right]$

and $L_S = L_q + C - \sum_{0}^{C-1}(C-n)P_n$

23**. Explain the term balking in the queue?**

It is one of the customer's behavior in the queue. A customer may leave the queue is very long of there is no waiting space. This is known as balking.

24.**Explain the terms Reneging, Priorities, Jockeying in the queue**.

Reneging: This occurs when the waiting customer leaves the queue due to impatience. **Priorities**: In certain applications, some customer served before others regardless of their order of arrival. **Jockeying**: Customers may jump from one waiting line to another for their personal gains.

25. **What is traffic intensity? If traffic intensity of M/M/A system is given to be 1.76, what percent of time the system would be idle?**

The server utilization factor or busy period or traffic intensity $\rho = \frac{\lambda}{\mu}$ is the proportion of time that a server actually spends with the customers. Given ρ = 0.76 expected idle time = 1- ρ = 1- 0.76 = 0.24 Idle time is 24 %

PART B Questions

1. A Petrol pump station has 2 pumps . The service times follow the exponential distribution with mean of 4 minutes and cars arrive for service is a Poisson process at the rate of 10 cars per hour. Find the probability that a customer has to wait for service. What is the probability that the pumps remain idle?
2. Obtain the steady state probability for (M/M/1) : (N/FCFS) queuing model
3. In a given M/M/1 queuing system, the average arrivals is 4 customers per minute and $\rho = 0.7$. What are (1) mean number of customers L_S (2) mean number of customers L_q in the queue (3) probability that the server is idle (4) mean waiting time W_S in the system
4. Automatic car wash facility operates with only one bay. Cars arrive according to a Poisson process, with mean of 4 cars per hour and may wait in the facility's parking lot if the bay is busy. If the service time for all cars is constant and equal to 10 min determine L_s, L_q, W_s and W_q
5. Derive the formula for the average number of customers in the queue and the probability that an arrival has to wait for (M/MC) with infinite capacity. Also derive for same model the average waiting time of a customer in the queue as well as in the system.
6. Customers arrive at a one man barber shop according to a Poisson process with a mean inter arrival time of 20 min. Customers spend an average of 15 minutes in the barber chair. If an hour is used as a unit of time, then (i) What is the probability that a customer need not wait for a hair cut? (ii) What is the expected number of customers in barbershop and in the queue? (iii) How much time can a customer spend in the queue? (iv) Find the average time that the customer spend in the queue?
7. A Concentrator receives messages from a group of terminal and transmits them over a sigle transmission line. Suppose that messages arrives according to a Poisson process at a rate of one message every 4 milliseconds and suppose that message transmission times are exponentially distributed with mean 3 ms. Find the mean number of messages in system and the mean total delay in the system. What percentage increased in arrival rate results in a dubling of the above mean total delay?
8. Patients arrive at a clinic according to a Poisson distribution at a rate of 30 patients per hour. The waiting room does not accommodate more than 14 patients. Examination time per patient is exponential with mean rate of 20 per hour. (i) What is the probability that an arriving patient will not wait? (ii) What is the effective arrival rate?
9. There are three typist in an office. Each typist can type an average of 6 letters per hour. If letters arrive for being typed at the rate of 15 letters per hour (i) What is the probability that no letters are there in the system? (ii) What is the probability that all the typists are busy?
10. Arrivals at a telephone booth are considered to be Poisson with an average time of 12 min between one arrival and the next. The length of a phone call is distributed exponentially wilth mean 4 minutes (i) What is the average number

of customers in the system? (ii) What fraction of the day the phone will be in use (iii) What us the probability that an arriving customer have to wait?

11. Derive the Pollaczek – Khinchine formula for M/G/1 queueing model
12. Customers arrive at a one man barber shop accoding to a Poisson process with a mean inter arrival time of 12 min. Cuatomers spend an average of 10 min in the barber's chair (i) What is the expected number of customers in the barber shop and in the queue? (ii) What is the probability that more than 3 customers are in system?
13. A super market has two girls attending sales at the counters. If the service time for each customer is exponential with mean 4 min and if people arrive in Poisson fashion at the rate of 10 per hour, (i) What is the probability that a customer has to wait for service? (ii) What is the expected percentage of idle time for each girl?

14. A dublicating machine maintained for office use is operated by an office assistant who earns Rs.5 per hour. The time to complete each jobs varies according to an exponential distribution with mean 6 mins. Assume a Poisson input with an average rate of 5 jobs per hour. If an 8 hours day is used as a base, determine (i) The percentage idle time of the machine (ii) The average time a job is in the system (iii) The average earning per day of the assistant
15. Obtain the expression for steady state probabilities of a M/M/C queuing system.
16. A repairman is to be hired to repair machines which breakdown at an average rate of 3 per hour. The breakdown follows Poisson distribution. Non productive time of machine is considered to cost Rs 16 per hour. Two repairman have been interviewed. One is slow but cheap while the other is fast and expensive. The slow repairman charges Rs 8 per hour and he services machines at the rate of 4 per hour. The fast repairman demands Rs 10 per hour and services at the average rate of 6 per hour. Which repairman should be hired?
17. A Two person barber shop has 5 chairs to accommodate waiting customers. Potential customers who arrive when all 5 chairs are full, leaving withour entering barber shop. Cusomers arrive at the average rate of 4 per hour and spend an average of 12 minutes in the barbers chair. Compute ρ_0 , ρ_τ and average number of customers in the queue
18. Derive the formula for (i) average numbers in the queue (ii) average waiting time of a customer in the queue for (M/M/1) : (∞/FIFO) model
19. on average 96 patients per 24 hour day require the service of an emergency clinic. Also on average a patient requires 10 minutes of active attention. Assume that the facility can hadle only one emergency at a time, suppose that it costs the clinic Rs 100 per patient treated to obtain an average servicing time of 10 minutes and that each minute of decrease in this average time would cost Rs 10 per patient treated. How much would have to be budgeted by the clinic to decrease the average sixe of the queue from $1\frac{1}{3}$ patient to $\frac{1}{2}$ patient?
20. A bank has two tellers working on savings account. The first teller handles withdrawals only. The second teller handles deposits only. If has been found

that the service time distributions for both deposits and withdrawals are exponential with mean service time of 3 minutes per customer. Depositors are found to arrive in a poisson fashion throught the day wilth mean arrival rate of 16 per hour. Withdrawers also arrive in a poisson fashion with mean arrival rate of 14 per hour. What would be the effect on the average waiting time for the cusftomers if each teller could handle both withdrawals and deposits.

21. In a heavy machine shop, the overhead crane is 75 % utilized. Time study observations gave the average slinging time as 10.5 minutes with a standard deviation od 8.8 minutes. What is the average calling rate for the services of the crane and what is average delay in getting service?. If the average service time is cut to 8.0 minutes, how much reduction will occur, on average, in the delay of getting served?

UNIT IV: ADVANCED QUEUEING THEORY

PART A

1. What is the probability then an arrival to an infinite capacity 3 server poisson queuing system with = 2μλ and Po= 1/9 enters the service without waiting.

Solution: P(without waiting) = P(N<3)= Po+P1+P2

P_n=

q λ*E W*

μ μ λ

λ

=

−

(iii)E(Ws)= E(Wq) +

μ

1

(iv)E(Ns) = E(Nq) +

μ

PART B

1 Derive the Balance equation of the birth and death process.

2 Derive the Pollaczek- Khinchine formula

3. Consider a single server, poisson input queue with mean arrival rate of 10 hour currently the server works according to an exponential distribution with mean service time of 5 minutes. Management has a training course which will result in an improvement in the variance of the service time but at a slight increase in the mean. After completion of the course;, it isestimated that the mean service time will increase to 5.5 minutes but the standard deviation will decrease from 5 minutes to 4 minutes. Management would like to know ;whether they should

have the server undergo further training.

4. In a heavy machine shop, the overhead crane is 75% utilized. Time study observations gave the average slinging time as 10.5 mInutes with a standard deviation of 8.8 minutes. What is the average calling rate for the services of the crane and what is the average delay in getting service? If the average service time is cut to 8.0 minutes, with a standard deviation of 6.0 minutes, how much reduction will occur, on average, in the delay of getting served?

5. Automatic car wash facility operates with only on Bay. Cars arrive according to a Poisson process, with mean of 4 cars per hour and may wait in the facility's parking lot if the bay is busy. If the service time for all cars is con;stant and equal to 10 min, determine Ls, Lq, Ws and Wq

<u>################################**ALL THE BEST** ######################</u>

Solution: Refer : Probability and Queueing Theory by G. Balaji

pg.5.59.

4.

Solution: Refer : Probability and Queueing Theory by G. Balaji

pg.5.61.

5.

Solution: Refer : Probability and Queueing Theory by G. Balaji

pg.5.64.

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